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On uniformly rotating fluid drops trapped between two parallel plates

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ABSTRACT. This contribution is about the dynamics of a liquid bridge between two fixed parallel plates. We consider a mathematical model and present some results from the doctorial thesis [10] of the first author. He showed that there is a Poisson bracket and a corresponding Hamiltonian, so that the model equations are in Hamiltonian form. The result generalizes previous results of Lewis et al. [12] on the dynamics of free boundary problems for "free" liquid drops to the case of a drop between two parallel plates, including, especially the effect of capillarity and the angle of contact between the plates and the free fluid surface. Also, we prove the existence of special solutions which represent uniformly rotating fluid bridges, and we present specific stability conditions for these solutions. These results extend work of Concus and Finn [2] and Vogel [18],[19] on static capillarity problems (see also Finn [5]). Using the Hamiltonian structure of the model equations and symmetries of the solutions, the stability conditions can be derived in a systematic way. The ideas that are desribed will be useful for other situations involving capillarity and free boundary problems as well.

1 Introduction

We consider the motion of an ideal, i.e., incompressible and inviscid fluid of finite volume between two parallel flat plates. The plates are assumed to be at rest. We only take into account surface tension along the free surface of the fluid and adhesion forces along the surfaces of contact between the fluid and the plates. The influence of other forces such as gravity is neglected (cf. Concus and Finn [3]). Also, the complete separation of the fluid from one of the plates

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will be excluded, i.e., we assume that the fluid always forms a bridge between the two plates.

In the theory of capillary phenomena, the boundary conditions that people use for the angle of contact of the liquid with the wall is still a point of controversy if the liquid is not at rest. In the latter case, Gauss [6] gave a justification of the particular boundary conditions that people use, based on the principle of virtual work. In the present situation, we argue that there is a distinguished natural choice for these boundary conditions dictated by the Hamiltonian structure. They can be derived as part of the variational principle underlying the Hamiltonian equations. In particular, this gives some justification for the assumption of a constant contact angle even in the dynamic case.

A standard way to find Hamiltonian structures for mechanical problems is to pass from the material to the spatial representation and derive the reduced Poisson structure (see e.g. Marsden et al. [15], [16]). With regard to this Poisson structure, the equations of motion read as follows

$$\dot{F} = \{F, H\} \quad \text{for all } F \in \mathcal{D}. \tag{1.1}$$

Here $\{\cdot,\cdot\}$ denotes the Poisson bracket on the reduced phase space $\mathcal{N}$, $\mathcal{D}$ is the class of smooth real-valued functions on $\mathcal{N}$, $\dot{F}$ denotes the derivative of F along solution curves, and $H \in \mathcal{D}$ is the corresponding Hamiltonian which describes the total energy. For fluid flow problems, this method dates back to Arnold [1] who considered pure rigid wall boundary conditions (cf. also Marsden and Weinstein [17]). Lewis et al. [12] have applied this method to a free boundary value problem in fluid dynamics for the first time. But this problem does not involve capillarity. The chief difficulty with free boundary value problems is the treatment of the boundary conditions. The situation is subtle, because the free boundary must be included as a dynamic variable. Thus, the Poisson bracket picks up boundary terms and this makes the interaction with the remaining terms subtle in terms of questions like the Jacobi identity. Because of this subtleties, one cannot just quote general reduction theory here. Rather one can only use this as a guide. For a dynamic problem involving capillarity, free surfaces and the contact angle as in the present case, things are even more subtle, because the curves of contact between the free fluid surface and the plates must be included amongst the dynamic variables in addition. For a derivation, along these lines, of the Hamiltonian structure which we are going to describe here, we refer to [10].

Uniformly rotating fluid drops are relative equilibrium solutions, in the shape of surfaces of revolution around an axis perpendicular to the plates. Our representation of these drop shapes in terms of Delauny curves determines them precisely for sufficiently small angular velocities. To analyze their stability, an energy method appropriate for rotating systems is used. If one restricts the stability analysis to rotationally symmetric initial perturbations of the given drop, then the definiteness of the relevant quadratic form can be determined using Sturmian theory which leads to specific stability criteria. In such a stability analysis, one has to be careful about the choice of a potential function, as there are several candidate potential energy functions that give the relative equilibria in terms of a variational principle. These make a difference, since the stability conditions need not be optimal if one does not make the best choice. The choice is also important for locating bifurcation points correctly (cf. Lewis [11], [13]). We indeed obtain the optimal stability conditions. Our choice of the potential function is suggested by the so-called reduced energy-momentum method which is a general method to test relative equilibria of Hamiltonian systems with rotational symmetries for their (nonlinear) stability (see e.g. Marsden [14, §5]). However, we also have to take into account the volume constraint since the fluid is assumed to be incompressible. The fact that the Hamiltonian system is infinite-dimensional in the present situation makes a rigorous stability analysis subtle in terms of questions like general existence and uniqueness of solutions (cf. Kröner [9]) or consistency of various topologies which are involved. We do not address such issues here. Rather we use a formal notion of stability (cf. Holm et al. [7]).

For static drops, some results along these lines were already known before (see Concus and Finn [2], and Vogel [18], [19]). For numerical results with rotation and even with gravity included, see Hornung and Mittelmann [8].

The following part of this paper is organized as follows. In section 2, we first state and discuss the basic equations of motion in conventional form. Then we describe their Hamiltonian structure, i.e., the corresponding Hamiltonian and Poisson bracket. In section 3, we outline the construction of uniformly rotating drop solutions by means of Delauny curves and formulate the stability criteria. As a simple example, we consider cylindrical drops.

2 The equations of motion

We assume that at any instant of time t, the free surface of the fluid drop is given as the graph of a real-valued function

$$r = \Sigma(\varphi, z; t) \qquad (r > 0,\ 0 \le \varphi < 2\pi,\ 0 \le z \le h), \tag{2.1}$$

where r, φ, z are cylindrical coordinates in the Euclidean 3-space with origin at one plate and the z-axis perpendicular to the plates; h denotes the distance of the two plates. Note that this assumption excludes the complete separation of the fluid drop from one of the plates. Then we have the following equations for spatial representations

$$v = v(r, \varphi, z; t), \quad p = p(r, \varphi, z; t), \quad \Sigma = \Sigma(\varphi, z; t)$$

of the velocity field, the pressure field and the free surface of the fluid:

$$\begin{array}{lll} \frac{\partial v}{\partial t} + (v \cdot \nabla)\, v = -\nabla p & \text{in} & D_\Sigma \\ \nabla \cdot v = 0 & \text{in} & D_\Sigma \\ \frac{\partial \Sigma}{\partial t} = \frac{v \cdot n}{e_r \cdot n} & \text{on} & r = \Sigma \\ p = \tau \kappa & \text{on} & r = \Sigma \\ v \cdot n = 0 & \text{on} & \Sigma_1 \cup \Sigma_2 \\ \cos \gamma_j = \frac{\sigma_j}{\tau} & \text{on} & c_j \quad (j = 1, 2) \end{array} \tag{2.2}$$

Here, ∇ is the Nabla operator; at any instant of time, D_Σ denotes the region between the plates which is occupied by the fluid; for $j = 1$ and $j = 2$, Σ_j is the region in the j-th plate P_j which is wetted by the fluid, and c_j denotes the boundary curve of Σ_j, i.e., the curve of intersection of the free surface $r = \Sigma$ with the plate P_j. The outer unit normal with respect to D_Σ is always denoted by n. The first two equations are just Euler's equations for ideal fluids, i.e., the balance equation for linear momentum and the continuity equation which models incompressibility of the fluid. The third equation constitutes a kinematic condition for the evolution of the free surface. Here and subsequently, e_r denotes the unit vector in radial direction. Along the free surface, the pressure is supposed to be balanced by surface tension. In the fourth equation, κ denotes the mean curvature of the free surface, and $\tau > 0$ is the material constant of surface tension. As usual, along the rigid walls, we assume slip boundary conditions given by the fifth equation. Finally, γ_j denotes the angle of contact of the fluid with the plate P_j, i.e., the angle between the outer unit normal n_j of Σ_j inside P_j and the outer unit conormal of the free surface $r = \Sigma$ along the curve c_j. It is assumed to be constant and given by the sixth equation in (2.2) where σ_j denotes the adhesion coefficient with respect to the plate P_j, which is another material constant. For simplicity, we have set the fluid density equal to one here.

As indicated in section 1, these equations are Hamiltonian in the sense of mechanics on Poisson manifolds. The *Hamiltonian function* is given by

$$(2.3)\quad H(\Sigma, v) = \frac{1}{2}\int_{D_\Sigma} \|v\|^2\, dV + \tau \int_\Sigma dA - \sum_{j=1,2} \sigma_j \int_{\Sigma_j} dA\,,$$

where the volume integral describes the kinetic energy of the fluid drop, $\|\cdot\|$ is the Euclidean norm, and the surface integrals describe the potential energies. The dynamic variables are the free boundary Σ and the spatial velocity field v, a divergence-free smooth vector field in the region D_Σ bounded by Σ and the plates P_j. Also, v is supposed to satisfy the slip boundary condition along Σ_1 and Σ_2. The surface Σ is represented by a sufficiently smooth function $\Sigma(\varphi, z)$ as in (2.1). We assume that the volume of D_Σ is prescribed. The (reduced) phase space $\mathcal{N}$ can be identified with all such pairs (Σ, v). Variations of Σ and v are denoted by $\delta\Sigma$ and δv, respectively. The Poisson bracket will be defined for functions $F, G : \mathcal{N} \to \mathbb{R}$, which possess functional derivatives defined as follows.

We say that such a function F has a *functional derivative with respect to* Σ at $(\Sigma, v) \in \mathcal{N}$, if there exist maps $\frac{\delta F}{\delta\Sigma}(\Sigma, v) : \Sigma \to \mathbb{R}$ and $\frac{\delta_j F}{\delta\Sigma}(\Sigma, v) : c_j \to \mathbb{R}$ ($j = 1, 2$), such that

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} F(\Sigma_\varepsilon, v) = \int_\Sigma \frac{\delta F}{\delta\Sigma}(\Sigma, v)\,\delta\Sigma\, dA + \sum_{j=1,2} \int_{c_j} \frac{\delta_j F}{\delta\Sigma}(\Sigma, v)\,\delta\Sigma\, ds$$

holds for any curve $\varepsilon \mapsto \Sigma_\varepsilon$ of admissible surfaces with $\Sigma_0 = \Sigma$ and $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \Sigma_\varepsilon = \delta\Sigma$. Here c_j denotes the curve of intersection of Σ with the plate P_j as above; for path integrals, the element of integration is denoted by ds. Similarly, we say that a function F has a *functional derivative with respect to* v at $(\Sigma, v) \in \mathcal{N}$, if there exists a divergence-free vector field $\frac{\delta F}{\delta v}(\Sigma, v)$ in D_Σ, the normal component of which vanishes along Σ_1 and Σ_2, such that

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} F(\Sigma, v_\varepsilon) = \int_{D_\Sigma} \frac{\delta F}{\delta v}(\Sigma, v)\delta v \, dV$$

holds for any curve $\varepsilon \mapsto v_\varepsilon$ of admissible vector fields with $v_0 = v$ and $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} v_\varepsilon = \delta_v$.

Let $\mathcal{D}$ be the set of all functions $F : \mathcal{N} \to \mathbb{R}$, which have functional derivatives as defined above at any point $(\Sigma, v) \in \mathcal{N}$. We have $H \in \mathcal{D}$. In fact,

$$\begin{aligned} \frac{\delta H}{\delta \Sigma}(\Sigma, v) &= (\frac{1}{2}\|v\|^2 + \tau\kappa)\, e_r \cdot n \\ \frac{\delta_j H}{\delta \Sigma}(\Sigma, v) &= (\tau \cos\gamma_j - \sigma_j)\, e_r \cdot n_j \quad (j = 1, 2) \\ \frac{\delta H}{\delta v}(\Sigma, v) &= v \quad . \end{aligned} \tag{2.4}$$

We now define a *Poisson bracket on* $\mathcal{N}$ as follows. For functions $F, G \in \mathcal{D}$, we set

$$\begin{aligned} \{F, G\} = & \int_{D_\Sigma} (\nabla \times v) \cdot (\frac{\delta F}{\delta v} \times \frac{\delta G}{\delta v})\, dV \\ & + \int_\Sigma \left[\frac{\delta F}{\delta \Sigma}(\frac{\delta G}{\delta v} \cdot n) - \frac{\delta G}{\delta \Sigma}(\frac{\delta F}{\delta v} \cdot n)\right] \frac{1}{e_r \cdot n}\, dA \\ & + \sum_{j=1,2} \int_{c_j} \left[\frac{\delta_j F}{\delta \Sigma}(\frac{\delta G}{\delta v} \cdot n) - \frac{\delta_j G}{\delta \Sigma}(\frac{\delta F}{\delta v} \cdot n)\right] \frac{1}{e_r \cdot n}\, ds. \end{aligned}$$

With this Poisson bracket, using the divergence theorem, it is not hard to show that for any solution (v, p, Σ) of the basic equation (2.2), the relation (1.1) is satisfied along the curve $t \mapsto (\Sigma, v)$ in $\mathcal{N}$. Conversely, given any such curve for which the relation (1.1) is satisfied, one can construct a pressure field p in D_Σ, such that (v, p, Σ) is a solution of (2.2) (see Kruse [10]). The pressure field satisfies the following boundary value problem, where Δ is the Laplace operator.

$$\begin{aligned} \Delta p &= -\nabla \cdot ((v \cdot \nabla) v) && \text{in } D_\Sigma \\ p &= \tau\kappa && \text{on } \Sigma \\ \nabla p \cdot n &= -((v \cdot \nabla) v) \cdot n && \text{on } \Sigma_j \ (j = 1, 2) \end{aligned} \tag{2.5}$$

In that sense, the equations in (2.2) are equivalent to the Hamiltonian equation (1.1).

3 Uniformly rotating liquid drops

Now we look for special solutions of (2.2), for which the fluid drops rigidly rotate around the z-axis with constant angular velocity ω, in the shape of surfaces of revolution

$$\Sigma = f(z) \quad (0 \le z \le h).$$

Obviously, such solutions are invariant under the group of rotations around the z-axis which is a symmetry group of the problem. With respect to the material representation, they are relative equilibria. For that kind of solutions, the basic equations reduce to the following boundary value problem for a nonlinear second-order ODE:

$$\begin{aligned} \tau\kappa &- \frac{1}{2}\omega^2 f^2 = c \quad (c \in \mathbb{R}) \\ f'(0) &= \frac{-\sigma_1}{\sqrt{\tau^2 - \sigma_1^2}} =: \rho_1 \\ f'(h) &= \frac{\sigma_2}{\sqrt{\tau^2 - \sigma_2^2}} =: \rho_2 \end{aligned} \tag{3.1}$$

Here, the mean curvature κ of the free surface is given by

$$\kappa = \kappa_f = \frac{1}{f(1+(f')^2)^{1/2}} - \frac{f''}{(1+(f')^2)^{3/2}}$$

in terms of f; f' and f'' are derivatives with respect to z, and c is an arbitrary real constant which is related to the pressure field as follows:

$$p = c + \frac{1}{2}\omega^2 r^2 \tag{3.2}$$

Note that, in terms of f, the volume of the drop is given by

$$vol(f) = \pi \int_0^h f^2\, dz. \tag{3.3}$$

Theorem 3.1 *Let the constants h, τ, ρ_1, ρ_2 as well as a number $K > 0$ be given. Then there exists a constant ω_0 such that for all $\omega \in [-\omega_0, \omega_0]$, (3.1) has a solution (f, c) with $vol(f) < K$.*

Proof: For $\omega = 0$, equation (3.1) says that the surface of revolution generated by f must have constant mean curvature. There is a simple construction of such functions f due to Delauny [4]. One rolls up an ellipse along the z-axis inside a plane without sliding. Then each focus describes a curve which generates a surface of constant mean curvature through rotation about the z-axis. Thus, using the parameters of the ellipse to fit the boundary conditions as well as the volume constraint, we can construct a solution f of (3.1) for $\omega = 0$. Furthermore, for small $|\omega| \neq 0$, we can use Delauny curves as a first approximation for the solution. Thus, the theorem follows by a perturbation argument.

To fit the boundary conditions, we use the shooting method. To this end, we consider the following initial value problem:

$$\begin{aligned} \tau\kappa_\phi \quad - \quad \frac{1}{2}\omega^2\phi^2 &= \frac{c}{r} \\ \phi(0) &= r\,f(0) \\ \phi'(0) &= \rho_1 \end{aligned} \tag{3.4}$$

Here r is a parameter, $r \in (0,1)$, and $c = c(f) = \tau\kappa_f$, where $f = f(z), z \in \mathbb{R}$, represents a Delauny curve given by the parametrization

$$f^2 = a^2(1 - 2e\cos\frac{s}{a} + e^2)\,.$$

Here

$$s = s(z) = \int_0^z (1 + f'(z)^2)^{\frac{1}{2}} dz + s_0\,, \quad s_0 \in \mathbb{R},$$

denotes the arc length parameter of the Delauny curve, a is the length of the major semiaxis of the underlying ellipse and e the numerical excentricity of this ellipse. Note that $0 \leq e < 1$ and $0 < f < 2a$. Also, f is a periodic function of z with period equal to the circumference of the underlying ellipse. Taking the derivative with respect to z in the formula for f^2, we immediately find

$$f' = e\,\frac{\sin\left(\frac{s}{a}\right)}{1 - e\cos\left(\frac{s}{a}\right)}\,.$$

This function is 2π-periodic and odd in $x = \frac{s}{a}$ and attains arbitrarily large positive and negative values near $x = 0$, as e approaches 1. Therefore, e can be chosen such that

$$\max_{\mathbb{R}} |f'| > \max(|\rho_1|, |\rho_2|)$$

holds. Then we have $f'(0) = \rho_1$ provided that the constant s_0 is chosen appropriately, and we can find values $z_0, z_1 > h$ such that $f'(z_0) > \rho_2$ and $f'(z_1) < \rho_2$. Without loss of generality, we assume that $z_0 > z_1$ holds. Finally we assume that a is chosen such that $4\pi\,h\,a^2 < K$ is satisfied.

Then a straightforward computation shows that for $\omega = 0$ and $r \in (0,1)$,

$$\phi(z) = r\,f(\frac{z}{r})\,, \quad z \in \mathbb{R},$$

is a solution of (3.4) such that in addition,the following inequalities hold:

$$\begin{aligned} \phi'(r\,z_0) &> \rho_2 \\ \phi'(r\,z_1) &< \rho_2 \\ vol(\phi) &< K \end{aligned} \tag{3.5}$$

Note that scaling f by r as above is equivalent to replacing a by ra. Finally we use the well known theorem of continuous dependence on parameters for the solution of an initial value problem such as (3.4) to argue that, in particular for $\frac{h}{z_0} \le r \le \frac{h}{z_1}$ and $0 \le z \le 2h$, there is a solution $\phi = \phi(z)$ of (3.4) and (3.5) even for sufficiently small $|\omega| \ne 0$, say for $|\omega| \le \omega_0$. This solution depends continuously on r, and by (3.5), $\phi'(h) = \phi'(rz_0) > \rho_2$ for $r = \frac{h}{z_0}$ and $\phi'(h) = \phi'(rz_1) < \rho_2$ for $r = \frac{h}{z_1}$. Hence, by the intermediate value theorem, for any $\omega \in [-\omega_0, \omega_0]$ there exists a value of $r \in [\frac{h}{z_0}, \frac{h}{z_1}]$ such that the corresponding function $f = \phi(z)$ solves the boundary value problem (3.1) for a certain value of $c \in \mathbb{R}$. This proves the theorem.

To analyze the stability of these special solutions, we use the *augmented potential function* $V_\omega : \mathcal{N} \to \mathbb{R}$ given by

$$V_\omega = V_p(\Sigma) - \frac{1}{2} I(\Sigma)\, \omega^2 ,$$

where V_p denotes the total potential energy of a drop (cf. 2.3), and $I(\Sigma) = \int_{D_\Sigma} r^2\, dV$ is its moment of inertia about the z-axis. For axially symmetric drop shapes given by a function $f(z)$, it follows that

$$\text{(3.6)} \quad V_\omega = V_\omega(f) \;=\; 2\pi\tau \int_0^h f\sqrt{1+(f')^2}\, dz \;-\; \sigma_1 \pi f(h)^2 \;-\; \sigma_2 \pi f(0)^2 \\ -\; \frac{\pi}{4}\omega^2 \int_0^h f^4\, dz \,.$$

Moreover, if f is a solution of (3.1), then f is a critical point of the functional $\tilde{V}_\omega = V_\omega - c\, vol$ on the function space $C^1[0, h]$ (see Kruse [10]), where the values of ω and c are the same as in (3.1). This modification of V_ω is consistent with the volume constraint $vol(f) = const.$.

The potential function V_ω is appropriate for our situation. In fact, let f be a solution of (3.1) and denote the corresponding surface of revolution by Σ_f and the velocity field corresponding to the angular velocity ω by v_ω. Define an *augmented Hamiltonian (energy-momentum functional)* on $\mathcal{N}$ by

$$H_\omega = H - \omega(J - J(\Sigma_f, v_\omega)) ,$$

where $J = J(\Sigma, v)$ is the momentum map that assigns to each drop state $(\Sigma, v) \in \mathcal{N}$ the corresponding angular momentum about the z-axis. This is a conserved quantity for our system due to the rotational symmetry. Hence, H_ω is also a conserved quantity. Furthermore, (Σ_f, v_ω) is a critical point of H_ω. Also, if f is a strict minimum of V_ω subject to the volume constraint, then (Σ_f, v_ω) is a strict minimum of H_ω restricted to the subset of pairs $(\Sigma, v) \in \mathcal{N}$, such that Σ is axially symmetric. Hence, H_ω is a kind of Liapunov function for (Σ_f, v_ω). This is a consequence of the fact that H_ω can be rewritten as

$$\text{(3.7)} \qquad H_\omega = K_\omega + V_\omega + \omega\, J(\Sigma_f, v_\omega)$$

with the *augmented energy functional* K_ω given by

$$K_\omega = \frac{1}{2} \int_{D_\Sigma} \|v - v_\omega\|^2\, dV \,;$$

cf. Marsden [14, §5]. By this reference, one expects to obtain sharper (optimal) stability conditions if instead of V_ω one works with the so-called *amended potential function* $V_\mu : \mathcal{N} \to \mathbb{R}$ given by

$$V_\mu = V_p(\Sigma) + \frac{1}{2} I(\Sigma)^{-1} \mu^2$$

with $\mu = J(\Sigma_f, v_\omega)$. However, as far as the stability analysis below is concerned, these two potential functions lead to the same results. Because V_μ is singular at $\Sigma = \Sigma_f$ with $f \equiv 0$, we have chosen to work with V_ω here.

These ideas motivate the following stability criterium. A solution (Σ_f, v_ω) is said to be *formally stable with respect to rotationally symmetric initial perturbations*, if the second variation of V_ω at $f \in C^1[0,h]$ is positive definite subject to the volume constraint, which amounts to saying (cf. Vogel [18]) that the quadratic form

$$\beta(\phi) = \int_0^h (P(\phi')^2 - Q(\phi)^2)\, dz \tag{3.8}$$

with

$$P = \frac{\tau h}{(1+(f')^2)^{3/2}} \quad , \quad Q = \frac{\tau}{f(1+(f')^2)^{1/2}} + \omega^2 f^2$$

is positive definite on the function space

$$C^1_\perp[0,h] = \left\{ \phi \in C^1[0,h] \;\middle|\; \int_0^h f\,\phi\, dz = 0 \right\}.$$

For $\omega = 0$, this stability criterium agrees with that in Vogel [18]. According to Vogel [18, Theorem 3.1], $\beta(\phi)$ is positive definite on $C^1_\perp[0,h]$, if the associated Sturm-Liouville eigenvalue problem

$$L[\phi] = \lambda\phi \quad , \quad \phi'(0) = \phi'(h) = 0 \tag{3.9}$$

with

$$L[\phi] = -(P\,\phi')' + Q\,\phi$$

has eigenvalues $\lambda_1 < 0 < \lambda_2 < \dots$, and the solution ψ of the boundary-value problem

$$L[\psi] = f \quad , \quad \psi'(0) = \psi'(h) = 0 \tag{3.10}$$

has the property $\int_0^h \psi\, dz < 0$. Thus, one can use Sturmian theory to verify the definiteness of the relevant quadratic form.

We conclude with a simple example. Suppose that $\sigma_1 = \sigma_2 = 0$ holds. Then $f(z) \equiv d$ is a solution of (3.1) for any ω, any $d > 0$ and a certain constant c. These solutions represent rigidly rotating cylindrical drops. In this case, the eigenvalues of the associated Sturm-Liouville eigenvalue problem (3.9) can be computed explicitly, since the operator $L[\phi]$ has constant coefficients. The smallest eigenvalue $\lambda_1 = -(\tau/d + \omega^2 d^2)$ is always negative. The other eigenvalues are given by $\lambda_{n+1} = \tau d n^2 \pi^2/h^2 - \tau/d - \omega^2 d^2$, $n \in \mathbb{N}$. Hence, the condition $\lambda_2 > 0$ leads to the explicit stability criterium

$$\frac{h^2}{\pi^2 d^2} + \frac{\omega^2 h^2 d}{\pi^2 \tau} < 1 \quad . \tag{3.11}$$

The solution $\psi = -\frac{d}{\tau/d + \omega^2 d^2}$ of (3.10) clearly has the required property. .

References

1. V.I. Arnold, Sur la géometrie differentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluids parfaits, Ann. Inst. Fourier, Grenoble **16** (1966), 319–361.

2. P. Concus and R. Finn, The shape of a pendent liquid drop, Philos. Trans. Roy. Soc. London Ser. A, **292** (1979), 307–340.

3. P. Concus and R. Finn, Capillary surfaces in microgravity, in *Low-Gravity Fluid Mechanics and Transport Phenomena*, J.N. Koster and R.L. Sani edts., Progress in Astronautics and Aeronautics **130** AIAA, Washington, DC, (1990), 183–206.

4. C. Delauny, J. Math. Pures Appl. **6** (1841), 309– .

5. R. Finn, *Equilibrium Capillary Surfaces*, Springer-Verlag, New York, 1986.

6. C.F. Gauss, *Werke (Collected Works)*, Vols. 1–12, Göttingen, 1863–1929.

7. D.D. Holm, J.E. Marsden, T. Ratiu and A. Weinstein, Nonlinear stability of fluid and plasma equilibria, Physics Reports **123** (1985), 1–116.

8. U. Hornung and H.D. Mittelmann, Bifurcation of axially symmetric capillary surfaces, J. Colloid and Interface Sci. **146** (1) (1991), 219–225.

9. D. Kröner, The flow of a fluid with a free boundary and dynamic contact angle, Z. angew. Math. Mech. **67** (1987), T 304–T 306.

10. H.-P. Kruse, *Flüssigkeitstropfen zwischen parallelen Platten: Hamiltonsche Struktur, Existenz von Lösungen und Stabilität*, Doctorial thesis, Universität Hamburg, 1992.

11. D.R. Lewis, Nonlinear stability of a rotating planar liquid drop, Arch. Rat. Mech. Anal. **106** (1989), 287–333.

12. D.R. Lewis, J.E. Marsden, R. Montgomery and T.S. Ratiu, The Hamiltonian structure for dynamic free boundary problems, Physica D **18** (1986), 391–404.

13. D.R.Lewis, J.E. Marsden and T.S. Ratiu, Stability and bifurcation of a rotating liquid drop, J. Math. Phys. **28** (1987), 2508–2515.

14. J.E. Marsden, *Lectures on Mechanics*, London Math. Soc. Lect. Note Ser. **174**, Cambridge University Press, Cambridge, 1992.

15. J.E. Marsden, T.S. Ratiu and A. Weinstein, Semi-direct products and reduction in mechanics, Trans. Am. Math. Soc. **281** (1984), 147–177.

16. J.E. Marsden, T.S. Ratiu and A. Weinstein, Reduction and Hamiltonian structures on duals of semidirect product Lie algebras, Cont. Math. AMS **28** (1984), 55–100.

17. J.E. Marsden and A. Weinstein, Coadjoint orbits, vortices and Clebsch variables for incompressible fluids, Physica D **7** (1983), 305–323.

18. T.I. Vogel, Stability of a liquid drop trapped between two parallel planes, SIAM J. Appl. Math. **47** (1987), 516–525.

19. T.I. Vogel, Stability of a liquid drop trapped between two planes II: general contact angles, SIAM J. Appl. Math. **49** (1989), 1009–1028.

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